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On the Diffraction of Light by Spheres of Small Relative Index.

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In a short paper "On the Diffraction of Light by Particles Comparable with the Wave-length,"* Keen and Porter describe curious observations upon the intensity and colour of the light transmitted through small particles of precipitated sulphur, while still in a state of suspension, when the size of the particles is comparable with, or decidedly larger than, the wave-length of the light. The particles principally concerned in their experiments appear to have decidedly exceeded those dealt with in a recent paper,† where the calculations were pushed only to the point where the circumference of the sphere is 2.25λ . The authors cited give as the size of the particles, when the intensity of the light passing through was a minimum, 6 to 10μ , that is over 10 wave-lengths of yellow light, and they point out the desirability of extending the theory to larger spheres.

The calculations referred to related to the particular case where the (relative) refractive index of the spherical obstacles is 1.5. This value was chosen in order to bring out the peculiar polarisation phenomena observed in the diffracted light at angles in the neighbourhood of 90° , and as not inappropriate to experiments upon particles of high index suspended in water. I remarked that the extension of the calculations to greater particles would be of interest, but that the arithmetical work would rapidly become heavy.

There is, however, another particular case of a more tractable character, viz., when the relative refractive index is *small*; and although it may not be

* 'Roy. Soc. Proc.,' A, vol. 89, p. 370 (1913).

† 'Roy. Soc. Proc.,' A, vol. 84, p. 25 (1910); 'Scientific Papers,' vol. 5, p. 547.

the one we should prefer, its discussion is of interest and would be expected to throw some light upon the general course of the phenomenon. It has already been treated up to a certain point, both in the paper cited and the earlier one* in which experiments upon precipitated sulphur were first described. It is now proposed to develop the matter further.

The specific inductive capacity of the general medium being unity, that of the sphere of radius R is supposed to be K , where $K-1$ is very small. Denoting electric displacements by f, g, h , the primary wave is taken to be

$$h_0 = e^{int} e^{ikx}, \quad (1)$$

so that the direction of propagation is along x (negatively), and that of vibration parallel to z . The electric displacements (f_1, g_1, h_1) in the scattered wave, so far as they depend upon the first power of $(K-1)$, have at a great distance the values

$$f_1, g_1, h_1 = \frac{h^2 P}{4\pi r} \left(\frac{\alpha\gamma}{r^2}, \frac{\beta\gamma}{r^2}, -\frac{\alpha^2 + \beta^2}{r^2} \right), \quad (2)$$

in which

$$P = -(K-1) \cdot e^{int} \iiint e^{ik(x-r)} dx dy dz. \quad (3)$$

In these equations r denotes the distance between the point (α, β, γ) where the disturbance is required to be estimated, and the element of volume ($dx dy dz$) of the obstacle. The centre of the sphere R will be taken as the origin of co-ordinates. It is evident that, so far as the secondary ray is concerned, P depends only upon the angle (χ) which this ray makes with the primary ray. We will suppose that $\chi = 0$ in the direction backwards along the primary ray, and that $\chi = \pi$ along the primary ray continued. The integral in (3) may then be found in the form

$$\frac{2\pi R^2 e^{-ikr}}{k \cos \frac{1}{2}\chi} \int_0^{\frac{1}{2}\pi} J_1(2kR \cos \frac{1}{2}\chi \cdot \cos \phi) \cos^2 \phi d\phi, \quad (4)$$

r now denoting the distance of the point of observation from the centre of the sphere. Expanding the Bessel's function, we get

$$P = -\frac{4\pi R^3 (K-1) e^{i(nt-kr)}}{3} \left\{ 1 - \frac{m^2}{2 \cdot 5} + \frac{m^4}{2 \cdot 4 \cdot 5 \cdot 7} - \frac{m^6}{2 \cdot 4 \cdot 6 \cdot 5 \cdot 7 \cdot 9} + \frac{m^8}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 5 \cdot 7 \cdot 9 \cdot 11} - \dots \right\}, \quad (5)$$

in which m is written for $2kR \cos \frac{1}{2}\chi$. It is to be observed that in this solution there is no limitation upon the value of R if $(K-1)^2$ is neglected absolutely. In practice it will suffice that $(K-1)R/\lambda$ be small, λ (equal to $2\pi/k$) being the wave-length.

* 'Phil. Mag.,' vol. 12, p. 81 (1881); 'Scientific Papers,' vol. 1, p. 518.

These are the formulæ previously given. I had not then noticed that the integral in (4) can be expressed in terms of circular functions. By a general theorem due to Hobson*

$$\int_0^{\frac{1}{2}\pi} J_1(m \cos \phi) \cos^2 \phi d\phi = \sqrt{\left(\frac{\pi}{2m}\right)} J_{\frac{3}{2}}(m) = \frac{\sin m}{m^2} - \frac{\cos m}{m}, \quad (6)$$

so that

$$P = -(K-1) \cdot 4\pi R^3 \cdot e^{i(nt-kr)} \left(\frac{\sin m}{m^3} - \frac{\cos m}{m^2} \right), \quad (7)$$

in agreement with (5). The secondary disturbance vanishes with P , viz., when $\tan m = m$, or

$$m = 2kR \cos \frac{1}{2}\chi = \pi(1.4303, 2.4590, 3.4709, 4.4747, 5.4818, \text{etc.}) \dagger \quad (8)$$

The smallest value of kR for which P vanishes occurs when $\chi = 0$, i.e. in the direction *backwards* along the primary ray. In terms of λ the diameter is

$$2R = 0.715\lambda. \quad (9)$$

In directions nearly along the primary ray *forwards*, $\cos \frac{1}{2}\chi$ is small, and evanescence of P requires much larger ratios of R to λ . As was formerly fully discussed, the secondary disturbance vanishes, independently of P , in the direction of primary vibration ($\alpha = 0, \beta = 0$).

In general, the intensity of the secondary disturbance is given by

$$f_1^2 + g_1^2 + h_1^2 = \left(\frac{k^2 P_0}{4\pi r} \right)^2 \left(1 - \frac{\gamma^2}{r^2} \right), \quad (10)$$

in which P_0 denotes P with the factor $e^{i(nt-kr)}$ omitted, and is a function of χ , the angle between the secondary ray and the axis of x . If we take polar co-ordinates (χ, ϕ) round the axis of x ,

$$1 - \frac{\gamma^2}{r^2} = 1 - \sin^2 \chi \cos^2 \phi; \quad (11)$$

and the intensity at distance r and direction (χ, ϕ) may be expressed in terms of these quantities. In order to find the effect upon the transmitted light, we have to integrate (10) over the whole surface of the sphere r . Thus

$$\begin{aligned} r^2 \iint \sin \chi d\chi d\phi (f_1^2 + g_1^2 + h_1^2) &= \pi \int_0^\pi \sin \chi d\chi \left(\frac{k^2 P_0}{4\pi} \right)^2 (1 + \cos^2 \chi) \\ &= \pi k^4 (K-1)^2 R^6 \int_0^\pi \sin \chi d\chi (1 + \cos^2 \chi) \frac{(\sin m - m \cos m)^2}{m^6} \\ &= \frac{1}{2} \pi k^2 R^4 (K-1)^2 \int_0^{2kR} \frac{dm}{m^5} \left\{ 2 - \frac{m^2}{k^2 R^2} + \frac{m^4}{4k^4 R^4} \right\} \\ &\quad \times \{1 + m^2 + (m^2 - 1) \cos 2m - 2m \sin 2m\}. \quad (12) \end{aligned}$$

* 'Lond. Math. Soc. Proc.,' vol. 25, p. 71 (1893).

† See 'Theory of Sound,' § 207.

The integral may be expressed by means of functions regarded as known
Thus on integration by parts

$$\begin{aligned} \int_0^m \{1 + m^2 + (m^2 - 1) \cos 2m - 2m \sin 2m\} \frac{dm}{m^5} \\ = -\frac{1 - \cos 2m}{4m^4} + \frac{\sin 2m}{2m^3} - \frac{1}{2m^2} + \frac{1}{2}, \\ \int_0^m \{1 + m^2 + (m^2 - 1) \cos 2m - 2m \sin 2m\} \frac{dm}{m^3} \\ = -\frac{1}{2m^2} + \int_0^m \frac{1 - \cos 2m}{m} dm + \frac{\cos 2m}{2m^2} + \frac{\sin 2m}{m} - 1, \\ \int_0^m \{1 + m^2 + (m^2 - 1) \cos 2m - 2m \sin 2m\} \frac{dm}{m} \\ = \int_0^m \frac{1 - \cos 2m}{m} dm + \frac{m^2}{2} + \frac{m \sin 2m}{2} + \frac{5 \cos 2m}{4} - \frac{5}{4}. \end{aligned}$$

Accordingly, if m now stand for $2kR$, we get

$$\begin{aligned} r^2 \iint \sin \chi d\chi d\phi (f_1^2 + g_1^2 + h_1^2) \\ = \frac{\pi R^2 (K-1)^2}{8} \left\{ -\frac{7(1 - \cos 2m)}{2m^2} - \frac{\sin 2m}{m} + 5 + m^2 \right. \\ \left. + \left(\frac{4}{m^2} - 4 \right) \int_0^m \frac{1 - \cos 2m}{m} dm \right\}. \quad (13) \end{aligned}$$

If m is small, the $\{ \}$ in (13) reduces to

$$0 + 0 \times m^2 + \frac{4}{27} m^4,$$

so that ultimately

$$(13) = \frac{8}{27} \pi k^4 R^6 (K-1)^2, \quad (14)$$

in agreement with the result which may be obtained more simply from (5).

If we include another term, we get

$$(13) = \frac{8}{27} \pi k^4 R^6 (K-1)^2 \left(1 - \frac{2k^2 R^2}{5} \right). \quad (15)$$

As regards the definite integral, still written as such, in (13), we have

$$\int_0^m \frac{1 - \cos 2m}{m} dm = \int_0^{2m} \left\{ \frac{x}{1 \cdot 2} - \frac{x^3}{4!} + \frac{x^5}{6!} - \dots \right\} dx = \gamma + \log(2m) - \text{Ci}(2m), \quad (16)$$

where γ is Euler's constant (0.5772156) and Ci is the cosine-integral, defined by

$$\text{Ci}(x) = \int_{\infty}^x \frac{\cos u}{u} du. \quad (17)$$

As in (16), when x is moderate, we may use

$$\text{Ci}(x) = \gamma + \log x - \frac{1}{2} \frac{x^2}{1 \cdot 2} + \frac{1}{4} \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \dots, \quad (18)$$

which is always convergent. When x is great, we have the semi-convergent series

$$\begin{aligned} \text{Ci}(x) = \sin x \left\{ \frac{1}{x} - \frac{1 \cdot 2}{x^3} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{x^5} - \dots \right\} \\ - \cos x \left\{ \frac{1}{x^2} - \frac{1 \cdot 2 \cdot 3}{x^4} + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{x^6} - \dots \right\}. \end{aligned} \quad (19)$$

Fairly complete tables of $\text{Ci}(x)$, as well as of related integrals, have been given by Glaisher.*

When m is large, $\text{Ci}(2m)$ tends to vanish, so that ultimately

$$\int_0^m \frac{1 - \cos 2m}{m} dm = \gamma + \log(2m).$$

Hence, when kR is large, (13) tends to the form

$$(13) = \frac{1}{2} \pi k^2 R^4 (K - 1)^2. \quad (20)$$

Glaisher's Table XII gives the maxima and minima values of the cosine-integral, which occur when the argument is an odd multiple of $\frac{1}{2}\pi$. Thus—

n .	$\text{Ci}(n\pi/2)$.	n .	$\text{Ci}(n\pi/2)$.
1	+0·4720007	7	-0·0895640
3	-0·1984076	9	+0·0700653
5	+0·1237723	11	-0·0575011

These values allow us to calculate the $\{ \}$ in (13), viz.,

$$-\frac{7(1 - \cos 2m)}{2m^2} - \frac{\sin 2m}{m} + 5 + m^2 + \left(\frac{4}{m^2} - 4 \right) [\gamma + \log 2m - \text{Ci}(2m)], \quad (21)$$

when $2m = n\pi/2$, and n is an odd integer. In this case $\cos 2m = 0$ and $\sin 2m = \pm 1$, so that (21) reduces to

$$-\frac{56}{n^2\pi^2} \pm \frac{4}{n\pi} + 5 + \frac{n^2\pi^2}{16} + \left(\frac{64}{n^2\pi^2} - 4 \right) [\gamma + \log(n\pi/2) - \text{Ci}(n\pi/2)]. \quad (22)$$

We find

n .	(22).	n .	(22).
1	0·0530	7	23·440
3	2·718	9	42·382
5	10·534	11	65·958

* 'Phil. Trans.,' vol. 160, p. 367 (1870).

For values of n much greater, (22) is sufficiently represented by $n^2\pi^2/16$, or m^2 , simply. It appears that there is no tendency to a falling-off in the scattering, such as would allow an increased transmission.

In order to make sure that the special choice of values for m has not masked a periodicity, I have calculated also the results when n is even. Here $\sin 2m = 0$ and $\cos 2m = \pm 1$, so that (21) reduces to

$$-\frac{112(1 \text{ or } 0)}{n^2\pi^2} + 5 + \frac{n^2\pi^2}{16} + \left(\frac{64}{n^2\pi^2} - 4\right) [\gamma + \log(n\pi/2) - \text{Ci}(n\pi/2)]. \quad (23)$$

The following are required :—

n .	$\text{Ci}(n\pi/2)$.	n .	$\text{Ci}(n\pi/2)$.
2	+0·0738	8	−0·0061
4	−0·0224	10	+0·0040
6	+0·0106		

of which the first is obtained by interpolation from Glaisher's Table VI, and the remainder directly from (19). Thus :—

n .	(23).	n .	(23).
2	0·7097	8	32·336
4	6·1077	10	53·477
6	16·156		

The better to exhibit the course of the calculation, the actual values of the several terms of (23) when $n = 10$ may be given. We have

$$-\frac{112}{n^2\pi^2} = -0·11348, \quad \frac{n^2\pi^2}{16} = 61·685,$$

$$4 - \frac{64}{n^2\pi^2} = 4 - 0·06485 = 3·93515,$$

$$\begin{aligned} \gamma + \log(\pi/2) + \log n - \text{Ci}(n\pi/2) &= 0·57722 + 0·45158 + 2·30259 - 0·0040 \\ &= 13·094, \end{aligned}$$

so that

$$\left(4 - \frac{64}{n^2\pi^2}\right) \{\gamma + \log(n\pi/2) - \text{Ci}(n\pi/2)\} = 13·094.$$

It will be seen that from this onwards the term $n^2\pi^2/16$, viz., m^2 , greatly preponderates; and this is the term which leads to the limiting form (20).

The values of $2R/\lambda$ concerned in the above are very moderate. Thus, $n = 10$, making $m = 4\pi R/\lambda = 10\pi/4$, gives $2R/\lambda = 5/4$ only. Neither

below this point, nor beyond it, is there anything but a steady rise in the value of (13) as λ diminishes when R is constant. *A fortiori* is this the case when R increases and λ is constant.

An increase in the light scattered from a single spherical particle implies, of course, a decrease in the light directly transmitted through a suspension containing a given number of particles in the cubic centimetre. The calculation is detailed in my paper "On the Transmission of Light through an Atmosphere containing Small Particles in Suspension,"* and need not be repeated. It will be seen that no explanation is here arrived at of the augmentation of transparency at a certain stage observed by Keen and Porter. The discrepancy may perhaps be attributed to the fundamental supposition of the present paper, that the relative index is very small, a supposition not realised when sulphur and water are in question. But I confess that I should not have expected so wide a difference, and, indeed, the occurrence of anything special at so great diameters as 10 wave-lengths is surprising.

One other matter may be alluded to. It is not clear from the description that the light observed was truly transmitted in the technical sense. This light was much attenuated—down to only 5 per cent. Is it certain that it contained no sensible component of scattered light, but slightly diverted from its original course? If such admixture occurred, the question would be much complicated.

* 'Phil. Mag.,' vol. 47, p. 375 (1899); 'Scientific Papers,' vol. 4, p. 397.